TMSP 2021/22
Problems 4 (04.04.2022)

1 Gaussian approximation

Evaluate the magnetization-magnetization correlation function $G(\vec{r}', \vec{r}''; T) = < \delta m(\vec{r}')\, \delta m(\vec{r}'') >$ within the gaussian approximation. What are the expressions for the correlation lengths for sub- and supercritical temperatures.

2 Spin waves

Consider the XY model on a d-dimensional hypercubic lattice with Hamiltonian

$$\mathcal{H}(\{\vec{s}_i\}) = -J \sum_{<ij>} \vec{s}_i \cdot \vec{s}_j,$$

where $|\vec{s}_i| = 1$, $i = 1, \ldots, N$, $N = L^d$, and $\vec{s}_i = (\cos\theta_i, \sin\theta_i)$. The partition function

$$Z(T, N) = \left(\prod_i \int d\vec{s}_i\right) e^{-\beta\mathcal{H}(\{\vec{s}_i\})} = \left(\prod_i \int d\theta_i\right) e^{K \sum\limits_{<ij>} \cos(\theta_i - \theta_j)}.$$

At low temperatures, i.e., for $K \gg 1$, an expansion around a ferromagnetically ordered ground state gives

$$Z(T, N) = e^{KNc} \left(\prod_i \int d\theta_i\right) e^{-\beta\mathcal{H}'} = e^{KNc} \left(\prod_i \int d\theta_i\right) e^{-\frac{K}{2} \sum\limits_{<ij>} (\theta_i - \theta_j)^2},$$

where $c = \mathit{const}$ and contributions from higher order terms have been neglected. This Hamiltonian can be diagonalized with the help of (discrete) Fourier transform

$$\{\theta_j\} \to \widetilde{\theta}_{\vec{q}} = \sum_j \theta_i\, e^{-i\vec{q}\cdot\vec{r}_j},$$

where periodic b.c. have been assumed ($\vec{q}$ takes disrete values). With the help of identity

$$\sum_j e^{-i(\vec{q}' - \vec{q}'')\cdot\vec{r}_j} = L^d\, \delta_{\vec{q}', \vec{q}''}$$

- show that

$$-\beta\mathcal{H}' = \frac{K}{2}\sum_{\vec{q}} \widetilde{\theta}_{\vec{q}}\,\widetilde{\theta}_{-\vec{q}} \sum_{\vec{e}} |1 - e^{i\vec{q}\cdot\vec{e}}\,|^2 = \frac{1}{2}\sum_{\vec{q}} K(\vec{q}\,)\,|\widetilde{\theta}_{\vec{q}}|^2$$

  What does the symbol $\vec{e}$ denote?

- show that for small q one obtains

$$K(\vec{q}) = K\,\vec{q}^{\,2}$$

- write down the expression for the partition function by performing integration over the spin-wave modes $(\widetilde{\theta}_{\vec{q}})$